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STRING THEORY IN LORENTZ-INVARIANT LIGHT CONE GAUGE

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Revised

Quantization of 4-dimensional Nambu-Goto theory of open string in light cone gauge, related in Lorentz-invariant way with the world sheet, is performed. Obtained quantum theory has no anomalies in Lorentz group. Determined spin-mass spectra of the theory have Regge-like behavior and do not contain the tachyon. Vertex operators of interaction theory, acting in the physical subspace, are constructed.

It is well known, that quantization of string theory creates anomalies, destroying classical symmetries of the system. Exception is a theory at critical value of space-time dimension $d = 26$, where the anomalies are absent. To construct 4-dimensional quantum string theory, required in physical applications, one usually introduces additional degrees of freedom (such as fermion fields, propagating along the world sheet), whose contribution cancels the anomaly at less number of dimensions. This approach can be combined with another idea: some of dimensions can be considered as coordinates on a compact manifold of physically negligible size.

There are also other methods of string quantization, which do not introduce extra dimensions or extra degrees of freedom, but use non-standard canonical bases in the phase space of Hamiltonian mechanics as a starting point for quantization. Such methods were applied in works [1,2] for quantization of special subsets in the phase space, representing particular types of the world sheets. Distinctive features of these 4-dimensional theories are absence of anomalies in Lorentz group and non-fixed intercept in spin-mass spectrum, which gives a possibility to remove tachyon from the theory. Due to these features, the given approach can be used for a construction of relativistic models of hadrons [3].

In this work we will try to extend the methods [1,2] for the world sheets of general form. For this purpose we use a definite modification of light cone gauge.

Light cone gauge relates a parametrization of the world sheet with some light-like vector (gauge axis), see fig.1. In standard approach this vector is non-dynamical, e.g. $n_\mu = (1, 1, 0, 0...)$. Because the Lorentz-transformations change the position of the world sheet respective to this fixed axis, they are followed by reparametrizations

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of the world sheet. On quantum level the reparametrization group has anomaly, which appears also in Lorentz group and violates Lorentz covariance of the theory. This is a main problem of string theory in standard light cone gauge.

The simple idea how to avoid this problem is to connect the gauge axis with some *dynamical* vector in string theory. In this case the Lorentz-transformations move the gauge axis together with the world sheet, and the parametrization on the world sheet is not changed. Lorentz group in this approach should be free of anomalies.

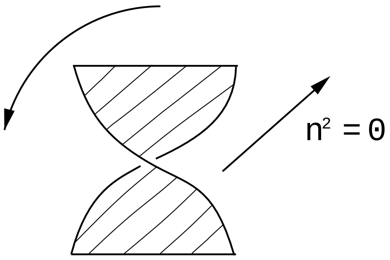

Fig.1. Light cone gauge.

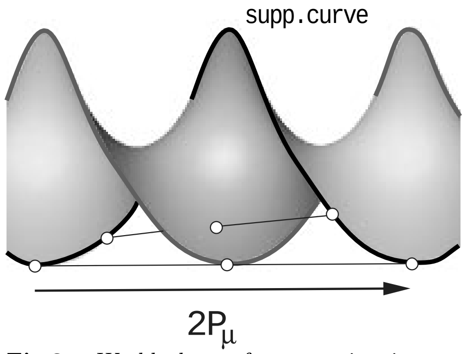

Fig.2. World sheet of open string is constructed as a locus of middles of segments, connecting all possible pairs of points on the supporting curve.

Implementation of this idea includes the following ingredients.

Geometrical description of the world sheet [4]. Let's introduce a function, related with string's coordinates and momenta by expressions

$$Q_\mu(\sigma) = x_\mu(\sigma) + \int_0^\sigma d\tilde{\sigma}\; p_\mu(\tilde{\sigma}), \tag{1}$$

$$x_\mu(\sigma) = (Q_\mu(\sigma) + Q_\mu(-\sigma))/2, \quad p_\mu(\sigma) = (Q'_\mu(\sigma) + Q'_\mu(-\sigma))/2 \tag{2}$$

(x, p are *even* functions of σ). In terms of oscillator variables, commonly used in string theory:

$$Q_\mu(\sigma) = X_\mu + \tfrac{P_\mu}{\pi}\sigma + \tfrac{1}{\sqrt{\pi}} \sum_{n\neq 0} \tfrac{a_\mu^n}{in} e^{in\sigma}. \tag{3}$$

The curve, defined by the function $Q_\mu(\sigma)$ (further called *supporting curve*) has the following properties:

1. the curve is light-like: $Q'^2(\sigma) = 0$, this property is equivalent to Virasoro constraints on oscillator variables;

2. the curve is periodical: $Q(\sigma + 2\pi) - Q(\sigma) = 2P$;

3. the curve coincides with the world line of one string end: $x(0,\tau) = Q(\tau)$; the world line of another end is the same curve, shifted onto the semi-period: $x(\pi,\tau) = Q(\pi+\tau) - P$;

4. the whole world sheet is reconstructed by this curve as follows: $x(\sigma,\tau) = (Q(\sigma_1)+Q(\sigma_2))/2$, $\sigma_{1,2} = \tau \pm \sigma$, see fig.2;

5. Poisson brackets for the function $Q_\mu(\sigma)$, and symplectic form, correspondent to these brackets (see Appendix 1):

$$\{Q_\mu(\sigma), Q_\nu(\tilde{\sigma})\} = -2g_{\mu\nu}\vartheta(\sigma-\tilde{\sigma}), \tag{4}$$
$$\Omega = \tfrac{1}{2}\, dP_\mu \wedge dQ_\mu(0) + \tfrac{1}{4}\int_0^{2\pi} d\sigma\ \delta Q'_\mu(\sigma) \wedge \delta Q_\mu(\sigma).$$

Here $\vartheta(\sigma) = [\sigma/2\pi] + \frac{1}{2}$, $[x]$ is integer part of x, a derivative $\vartheta(\sigma)' = \Delta(\sigma)$ is periodical delta-function.

These properties can be easily proven from definition of $Q_\mu(\sigma)$ and known mechanics in oscillator variables, see Appendix 2.

Mechanics in center-of-mass frame [1]**.** Let's introduce orthonormal tetrad of vectors, dependent on total momentum: $N^\alpha_\mu(P)$, $N^\alpha_\mu N^\beta_\mu = g^{\alpha\beta}$, with $N^0_\mu = P_\mu/\sqrt{P^2}$. Let's decompose the supporting curve by this tetrad: $Q_\mu(\sigma) = N^\alpha_\mu Q^\alpha(\sigma)$.

Lorentz-invariant light cone gauge [2]**.** Virasoro constraints generate reparametrizations of supporting curve (see Appendix 2). Gauges to Virasoro constraints select particular parametrization on this curve.

Let's use a parametrization: $Q^\alpha(\sigma) = Q^\alpha(0) + \int_0^\sigma d\sigma' a^\alpha(\sigma')$,

$$\begin{aligned} a^\alpha(\sigma) \;\; = \;\; & \left(\tfrac{\pi}{2\sqrt{P^2}}\left(\tfrac{P^2}{\pi^2} + |a(\sigma)|^2\right),\ \tfrac{a(\sigma)+a^*(\sigma)}{2}\,\mathbf{e}_1 + \tfrac{a(\sigma)-a^*(\sigma)}{2}\, i\mathbf{e}_2 \right. \\ & \left. + \tfrac{\pi}{2\sqrt{P^2}}\left(\tfrac{P^2}{\pi^2} - |a(\sigma)|^2\right)\mathbf{e}_3\right), \end{aligned} \tag{5}$$

where $a(\sigma) = \sqrt{\frac{2}{\pi}}\sum\limits_{n\neq 0} a_n e^{-in\sigma}$ and $\mathbf{e}_k$ is an orthonormal basis in CMF. Here one easily recognizes the light cone gauge $Q^+(\sigma) \equiv n^-_\mu Q_\mu(\sigma) = Q^+(0) + \frac{1}{\pi}\sqrt{\frac{P^2}{2}}\,\sigma$ with the gauge axis $n^-_\mu = \frac{1}{\sqrt{2}}(N^0_\mu - N^i_\mu e^i_3)$. The difference from standard approach is that n^-_μ is now *dynamical* vector, because we interpret $\mathbf{e}_k$ as dynamical variables.

Substituting this parametrization into general symplectic form (4), we obtain

$$\Omega = dP_\mu \wedge dZ_\mu \; + \sum_{k\neq 0} \tfrac{1}{ik}\, da^*_k \wedge da_k \; + \; \tfrac{1}{2}\, d\mathbf{e}_i \wedge d(\mathbf{S}\times\mathbf{e}_i), \tag{6}$$

where $Z_\mu = \frac{1}{2\sqrt{P^2}}\int\limits_0^{2\pi} d\sigma a^0(\sigma)\left(Q_\mu(\sigma) - \left(\frac{\sigma}{\pi} - 1\right)P_\mu\right) + \frac{1}{2}\epsilon^{ijk}\Gamma^{ij}_\mu S^k$ is mean coordinate, conjugated to P_μ; $\Gamma^{ij}_\mu = N^i_\nu \partial N^j_\nu/\partial P_\mu$ – Christoffel symbols and $\mathbf{S} =$

$-\frac{1}{4}\int\limits_0^{2\pi} d\sigma \int\limits_0^{\sigma} d\sigma' \ \mathbf{a}(\sigma)\times\mathbf{a}(\sigma')$ is an orbital moment of the string in CMF (further called spin). The last expression can be written in terms of oscillators a_k and can be interpreted as *constraints* :

$$\begin{aligned}
&\chi_3 = S_3 - A_3 = 0, \quad \chi^+ = S^+ - A^+ = 0, \quad \chi^- = S^- - A^- = 0, \quad \text{where} \\
&S_i = S^k e_i^k \text{ is a projection of } \mathbf{S} \text{ onto } \mathbf{e}_i;\ S^{\pm} = S_1 \pm iS_2;\ \chi^{\pm} = \chi_1 \pm i\chi_2; \\
&A_3 = \sum_{n\neq 0} \tfrac{1}{n} a_n^* a_n, \quad A^- = \sqrt{\tfrac{2\pi}{P^2}} \sum_{k,n,k+n\neq 0} \tfrac{1}{k} a_k a_n a_{k+n}^*, \quad A^+ = \text{ c.c.}
\end{aligned} \tag{7}$$

Another constraint is given by the requirement of $2P$-periodicity of the curve:

$$2P_\mu = \int_0^{2\pi} d\sigma \ a_\mu(\sigma) \ \Leftrightarrow \ \chi_0 = \tfrac{P^2}{2\pi} - L_0 = 0, \quad L_0 = \sum_{n\neq 0} a_n^* a_n.$$

The obtained symplectic form corresponds to the following Poisson brackets:

$$\begin{aligned}
&\{Z_\mu, P_\nu\} = g_{\mu\nu},\ \{a_k, a_n^*\} = ik\delta_{kn},\ k,n \in \mathbf{Z}/\{0\}, \\
&\{S^i, S^j\} = -\epsilon^{ijk} S^k,\ \{S^i, e_n^j\} = -\epsilon^{ijk} e_n^k.
\end{aligned} \tag{8}$$

Algebra of constraints belongs the 1st class: $\{\chi_0, \chi_i\} = 0$, $\{\chi_i, \chi_j\} = \epsilon_{ijk}\chi_k$.

Thus, the string is equivalent to a mechanical system:

$$P_\mu, Z_\mu + \text{infinite set of oscillators } a_k, a_k^* + \text{the top } \mathbf{e}_i, \mathbf{S},$$

restricted by 4 constraints of the 1st class, which include mass shell condition and requirements of the form "spin of the top is equal to the spin of the string". Constraints generate reparametrizations of the supporting curve: χ_0 generates shift of argument $Q(\sigma) \to Q(\sigma+\tau)$ (evolution of the string, see Appendix 2); χ_i generate the rotations of basis $\mathbf{e}_i$ with respect to non-moving supporting curve.

In more details: χ_3 rotates $\mathbf{e}_1$ and $\mathbf{e}_2$ about $\mathbf{e}_3$ and simultaneously rotates coefficients of supporting curve decomposition in opposite direction, so that supporting curve is not changed: $\{\chi_3, Q(\sigma)\} = 0$. Constraints $\chi_{1,2}$ generate the rotations of basis, changing the direction of the gauge axis n_μ^-, and they are followed by reparametrization of the supporting curve. In the oscillator variables this reparametrization looks like a complicated nonlinear transformation.

Generators of Lorentz group are defined by expression [1]

$$\begin{aligned}
&M_{\mu\nu} = \int_0^{\pi} d\sigma (x_\mu p_\nu - x_\nu p_\mu) = X_\mu P_\nu - X_\nu P_\mu + \epsilon_{ijk} N_\mu^i N_\nu^j S^k, \\
&X_\mu = Z_\mu - \tfrac{1}{2}\epsilon_{ijk}\Gamma_\mu^{ij} S^k,
\end{aligned}$$

they generate Lorentz transformations of a coordinate frame $(N_\mu^0, N_\mu^k e_i^k)$, by which the configuration is decomposed with scalar coefficients. Thus, $M_{\mu\nu}$ generate "rigid" Lorentz transformations of the world sheet, not changing its parametrization. Lorentz generators are in involution with constraints: $\{M_{\mu\nu}, \chi_{0,i}\} = 0$.

Lorentz generators are simple functions of variables $(Z, P, \mathbf{S})$, which in our approach are independent, i.e. their quantum commutators are postulated directly from Poisson brackets. As a result, in quantum mechanics the commutators $[M_{\mu\nu}, M_{\rho\sigma}]$, $[M_{\mu\nu}, Q_\rho]$ *are anomaly free.* This can be proven by direct calculation, done in [1].

The algebra of constraints χ_i in quantization acquires the same anomaly that earlier was in Lorentz group. Thus, our current result is just a transfer of anomaly from Lorentz group to the algebra of constraints. However, now we have more freedom to solve the problem, because we can impose additional gauges, excluding anomalous component in the algebra of constraints. Gauges relate the position of gauge axis with other dynamical vectors in the system.

Gauge 1: let's direct $\mathbf{e}_3 \uparrow\uparrow \mathbf{S} \;\Leftrightarrow\; S_1 = 0,\ S_2 = 0$. These gauges are in week involution with $\chi_{0,3}$, and are the gauges only for $\chi_{1,2}$. They are equivalent to 2nd class constraints onto oscillator variables: $A^\pm = 0,\ \{A^+, A^-\} = 2iS \neq 0$. The reduction of oscillator symplectic form onto the surface of these constraints leads to a form of complicated structure. This structure becomes simple for a definite restricted class of configurations.

Configurations with axial symmetry. Let's consider supporting curves, whose projection to CMF has axial symmetry of order 2 (see fig.3).

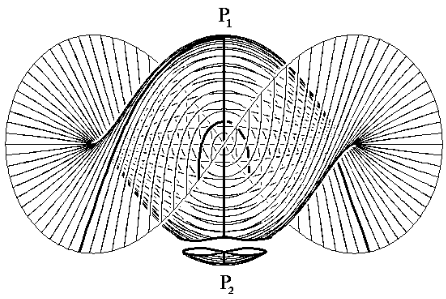

Fig.3. Supporting curve and the world sheet with axial symmetry of order 2 (both are projected to CMF). Computer generated image [5].

This restriction is equivalent to annulation of all even oscillator variables: $a_{2n} = 0,\ a^*_{2n} = 0,\ n \in \mathbf{Z}$. On the surface of this restriction the constraints $A^\pm = 0$ are satisfied *identically*, because all terms in the sum $\sum \frac{1}{k} a_k a_n a^*_{k+n}$ vanish (if k, n are odd, then $k+n$ is even). Reduction on the surface of all constraints will give Poisson brackets:

$$\begin{aligned} &\{Z_\mu, P_\nu\} = g_{\mu\nu},\ \{a_k, a^*_n\} = ik\delta_{kn}, \\ &\{S^i, S^j\} = -\epsilon^{ijk} S^k,\ \{S^i, e^j_1\} = -\epsilon^{ijk} e^k_1, \end{aligned} \tag{9}$$

the difference from the general case (8) is that indices of oscillator variables here are odd, and $(\mathbf{S}, \mathbf{e}_1)$ represents the mechanics of the rotator: $\mathbf{S}\mathbf{e}_1 = 0,\ (\mathbf{e}_1)^2 = 1$ (instead of the top in general mechanics). Remaining constraints: $\chi_0 = P^2/2\pi - L_0,\ \chi_3 = S - A_3$ are of the 1st class. As in general mechanics, χ_0 generates the evolution $Q(\sigma) \to Q(\sigma + \tau)$ and χ_3 does not change the configuration: $\{\chi_3, Q(\sigma)\} = 0$.

Quantization of this mechanics is straightforward. Canonical operators

$$[Z_\mu, P_\nu] = -ig_{\mu\nu},\ [a_k, a_n^+] = k\delta_{kn},\ k, n \text{ odd},$$
$$[S^i, S^j] = i\epsilon^{ijk}S^k,\ [S^i, e_1^j] = i\epsilon^{ijk}e_1^k,\ \mathbf{S}\mathbf{e}_1 = 0,\ (\mathbf{e}_1)^2 = 1$$

can be realized in a direct product of Fock space (with a vacuum $a_k|0\rangle = 0,\ k > 0,$ $a_k^+|0\rangle = 0,\ k < 0$)[a] onto the space of functions $\Psi(P, \mathbf{e}_1)$, with definition of operators $Z = -i\partial/\partial P,\ \mathbf{S} = -i\mathbf{e}_1 \times \partial/\partial \mathbf{e}_1$. Physical subspace is defined by 2 constraints $\left(\frac{P^2}{2\pi} - \sum\limits_{odd\ k} |k|n_k - \delta\right)|\Psi\rangle = 0,\ \left(S - \sum\limits_{odd\ k} \text{sign } k \cdot n_k\right)|\Psi\rangle = 0$ (here $\delta > 0$ is arbitrary c-number). Spin-mass spectrum of this mechanics is shown on fig.4. It's also possible to construct the operator $Q_\mu(\sigma)$, which satisfies all necessary requirements: is finite, Hermitian and commutators $[Q(\sigma), \chi_{0,3}]$ repeat classical Poisson brackets (see [2] for details).

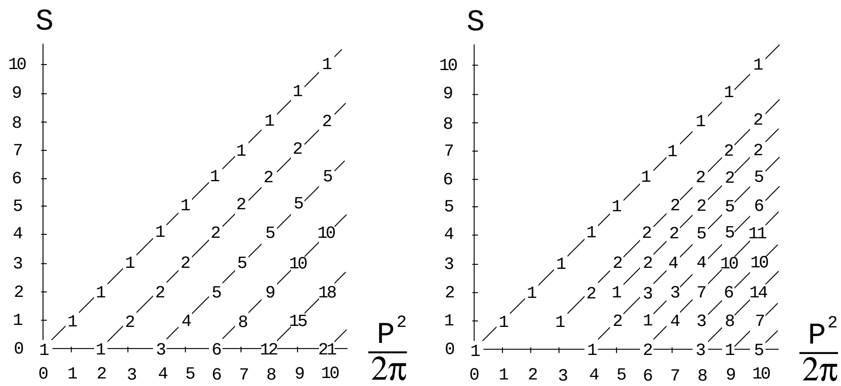

Fig.4. On the left: spectrum of axially symmetrical configurations; on the right: spectrum of general configurations in Gupta-Bleuler's approach; ($\delta \to +0$).

<u>*Remark:*</u> general configurations in Gupta-Bleuler's approach. We considered the mechanics of restricted configurations. For general configurations one should deal with a complicated mechanics, appearing in the reduction on the 2nd class constraints $A^\pm = 0$. However, it's possible to construct self-consistent quantum theory by methods, deviating from standard Dirac's procedure. For example, let's replace two 2nd class constraints by a single constraint: $A^\pm = 0\ (A_{1,2} = 0)\ \to\ A^- A^+ = A_1^2 + A_2^2 = 0$. In quantum theory we will have $A^- A^+ |\Psi\rangle = 0\ \Rightarrow\ \langle\Psi|A^- A^+|\Psi\rangle = 0$, and *in positively defined space* we are using this is equivalent to $A^+|\Psi\rangle = 0$ – the result coincides with Gupta-Bleuler's imposition of Hermitian conjugated constraints. Additionally we have the constraints $(S - A_3)|\Psi\rangle = 0,\ (P^2/2\pi - L_0 - \delta)|\Psi\rangle = 0$, the whole set of constraints belongs to the 1st class. Moreover, because $L_0 = \sum |k|n_k$, at any fixed value of $P^2/2\pi$ the mass shell condition defines *finite-dimensional* subspace, where other constraints act as matrices of finite size. It's easy to resolve the

[a] Such Fock space is positively defined, and occupation number operators $n_k = :a_k^+ a_k:/|k|$ take values: 0,1,2...

correspondent linear systems (see Appendix 3) and find the spectrum, shown on fig.4, right.

It's necessary to understand, that the described method is *different* from Dirac's approach, where imposition of constraint $\chi|\Psi\rangle = 0$ should imply, that physical states are invariant under the action of a gauge transformations, generated by χ. In our case the replacement of two 2nd class constraints by their sum of squares gives *a degenerate* constraint, which generates no transformations.

This method of constraints imposition has some physical ground. Let's consider a simplest example of 2nd class constraints: $x = 0,\ p = 0$. The Dirac's reduction completely eliminates these 2 degrees of freedom. If we replace these constraints by a single one: $x^2 + p^2 = a^*a = 0$, after an appropriate choice of quantum ordering we will have a constraint $a^+a|\Psi\rangle = 0$, defining the *vacuum* state, where x, p are distributed by Gaussian function with the dispersion $\hbar$. Taking $\hbar \to 0$, we will have the same situation, as in classical theory.

Realization of *2nd class* constraints by a single constraint a'la Gupta-Bleuler is widespread in literature (example is the method of string quantization, which was proposed by Rohrlich [6] and then was extensively used for a construction of hadrons' models, see [7]). We, however, want to use purely Dirac's methods and now will try to quantize the theory in another gauge.

Gauge 2, general configurations: let's direct $\mathbf{e_3} \perp \mathbf{S}\ \Leftrightarrow\ S_3 = 0$. Replacing the constraints $S^\pm - A^\pm = 0$ by equivalent combination $|S^+| - |A^+| = 0,\ \arg S^+ - \arg A^+ = 0$, we see that $S_3 = 0$ is a gauge only to the last constraint (it generates phase rotations of S^+), and is in involution with others.

Reduction leads to a mechanics, different from general one (8) by a replacement: the top $\mathbf{e}_i, \mathbf{S}\ \to$ the rotator $\mathbf{e}_3, \mathbf{S}$. Mechanics includes three 1st class constraints: $P^2/2\pi - L_0 = 0,\ A_3 = 0,\ S - \sqrt{A^+A^-} = 0$. First two constraints generate phase rotations of oscillator variables: $a_n \to a_n e^{-in\tau},\ a_n \to a_n e^{-i\tau}$, and A^+A^- is a polynomial of a_n, a_n^* with such structure, that it is conserved in these phase rotations. Because this property can be easily preserved in quantization, the quantum algebra of constraints will be free of anomalies. Configuration $a^\alpha(\sigma)$ has a form:

$$\begin{aligned} a^\alpha(\sigma) \;&=\; \Big(\tfrac{\pi}{2\sqrt{P^2}}\left(\tfrac{P^2}{\pi^2} + |a(\sigma)|^2\right), \tfrac{1}{2S}(a(\sigma)A^+\mathbf{n} + \text{ c.c. }) \\ &\quad + \tfrac{\pi}{2\sqrt{P^2}}\left(\tfrac{P^2}{\pi^2} - |a(\sigma)|^2\right)\mathbf{e}_3\Big), \end{aligned} \tag{10}$$

where $\mathbf{n} = (\mathbf{S} - i\mathbf{S}\times\mathbf{e}_3)/S$. As earlier, mass shell condition generates the evolution $Q(\sigma) \to Q(\sigma+\tau)$ and $\{A_3, Q(\sigma)\} = 0$ (A_3 rotates the phases of $a(\sigma)$ and A^+ in opposite directions and conserve $a(\sigma)A^+$). The constraint $S - \sqrt{A^+A^-} = 0$ generates the rotations of gauge axis about spin vector, and correspondent reparametrizations of the supporting curve.

P-reflection operation can be defined in this mechanics, consisting of two factors:
Π : reflection of the supporting curve w.r.t. a plane, perpendicular to $\mathbf{S}$, which is performed by a replacement $a_n \to a^*_{-n}\ \Rightarrow\ a(\sigma) \to a^*(\sigma),\ A^+ \to -A^-$;

$R:$ rotation of the supporting curve about spin with angle π: $\mathbf{e}_3 \to -\mathbf{e}_3,\ \mathbf{S} = Const \ \Rightarrow\ \mathbf{n} \to \mathbf{n}^*$;
so that $\mathbf{a}(\sigma) \to -\mathbf{a}(\sigma)$.

Remark: in gauge 1 the gauge axis was directed along the spin, which is not changed in P-reflection. Thus, the gauge axis is not changed also. As a result, P-reflection changes a position of the supporting curve w.r.t. gauge axis and is followed by reparametrization. This makes problematic the definition of P-reflection in gauge 1.

Quantization. Canonical operators:

$$[Z_\mu, P_\nu] = -ig_{\mu\nu},\ [a_k, a_n^+] = k\delta_{kn},\ k,n \in \mathbf{Z}/\{0\},\ [S^i, S^j] = i\epsilon^{ijk}S^k,\ [S^i, e_3^j] = i\epsilon^{ijk}e_3^k,$$

where $\mathbf{S}\mathbf{e}_3 = 0,\ (\mathbf{e}_3)^2 = 1$. Realization is analogous to presented above. Operators $A^\pm$, defined by the same polynomial expressions as in (7), have no ordering ambiguities. Three constraints:

$$(P^2/2\pi - L_0 - \delta)|\Psi\rangle = 0, \quad A_3|\Psi\rangle = 0, \quad \left(S - \sqrt{(A^+A^- + A^-A^+)/2}\ \right)|\Psi\rangle = 0$$

belong to the 1st class. Symmetric ordering under the square root in the last constraint was chosen, because it commutes with P-reflection, defined as follows:

$$P = R\Pi,\ R = e^{i\pi S},\ \Pi = \{a_k \to a_{-k}^+\} = \{|n_k\rangle \to |n_{-k}\rangle\}$$
$$\Rightarrow\ \Pi A^+\Pi = -A^-,\ \Pi A^+A^-\Pi = A^-A^+.$$

Again, at fixed $P^2/2\pi$ the mass shell condition defines finite-dimensional subspace, where other constraints act as finite matrices. It's easy to solve the eigenvalue problem for these matrices (see Appendix 3).

The main obstacle now is that operators $A^\pm$ still have the anomaly in commutator. If they would not have anomaly, the definition of the square root $T = \sqrt{(A^+A^- + A^-A^+)/2}$, analogous to $S = \sqrt{\mathbf{S}^2 + 1/4} - 1/2 \ \Leftrightarrow\ \mathbf{S}^2 = S(S+1)$ will give integer spectrum for T. But now $A^\pm$ have anomaly and do not represent the rotation group. The spectrum of T is not integer (even in S-like definition), see fig.5. Due to the constraint $S - T = 0,\ S \in \mathbf{Z},\ T \notin \mathbf{Z}$ the theory becomes empty.

This problem (anomaly in spectrum) is absolutely different from the usual one (anomaly in commutator). The following solution can be proposed.

Generally we can add to T any operator, which commutes with all constraints, and whose contribution is classically vanishing (e.g. eigenvalues of δT are bounded by Planck's constant: $|\delta T_i| < Const \cdot \hbar$). The corrected variable will have the same classical limit as T. With the aid of these corrections we can *deform* the spectrum of T to integer values. There is an infinite number of possible deformations, the simplest one: shift eigenvalues of T to the closest integer value below. Because we don't change the eigenvectors of T, the constraints remain to be of the 1st class (T acts in the subspaces, defined by other constraints). Redefinition changes T by operator, whose eigenvalues are restricted between 0 and 1, or when we restore Planck's constant – between 0 and $\hbar$. Therefore, the corrected operator T has the same classical limit[b]. After the redefinition $T \to [T]$ we have spin-mass spectrum, shown on fig.5, right.

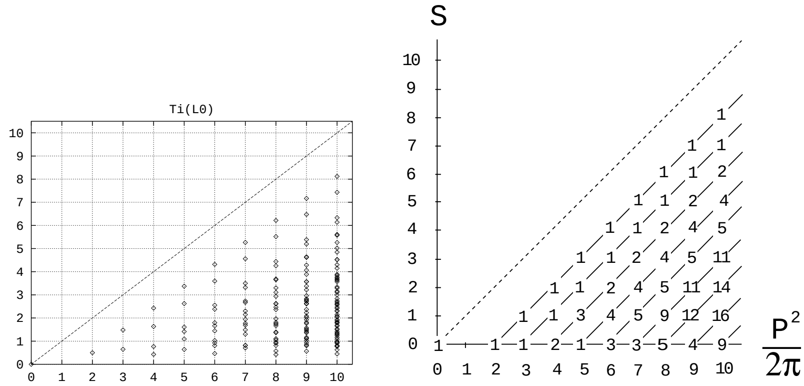

Fig.5. On the left: spectrum $T(L_0)$; on the right: integer part of this spectrum.

<u>*Remark*</u>: Leading Regge trajectory (starting from the vacuum state $L_0 = 0$) is seemingly absent in the spectrum. This effect, of course, can be explained as anomaly-induced nonlinearity of the leading trajectory $T_i(L_0)$ (fig.5, left). However, there are arguments, why the leading trajectory really should be absent in this approach, see Appendix 4.

Vertex operators. The operator $a^\alpha(\sigma)$ has ambiguities in ordering. A possible definition is:

$$a^\alpha(\sigma) \quad = \quad \left(\tfrac{1}{\sqrt{P^2}} \left(\tfrac{P^2}{\pi} + \sum_{n\neq 0} L_n e^{-in\sigma} \right) , \; \left(a(\sigma) A^+ \tfrac{1}{2S+1} \mathbf{n} + \text{ h.c.} \right) \right. \tag{11}$$

$$\left. - \tfrac{1}{\sqrt{P^2}} \sum_{n\neq 0} L_n e^{-in\sigma} \mathbf{e}_3 \right) .$$

Here $L_n = \sum a_k a_{k-n}^+$ and $\mathbf{n} = \frac{1}{S+1/2}(\mathbf{S} - i\mathbf{S}\times\mathbf{e}_3 - \mathbf{e}_3/2)$, see [8]. Configuration operator is finite[c], Hermitian and satisfies the relations: $[a^\alpha(\sigma), L_0] = ia^\alpha(\sigma)'$, $[a^\alpha(\sigma), A_3] = 0$.

Anomaly can appear in commutator $[a^\alpha(\sigma), S - T]$[d]. In classical mechanics the Poisson bracket $\{a^\alpha(\sigma), S - T\}$ is a complicated non-linear expression of independent variables, and we have the ordering ambiguity in definition of correspondent quantum expression. In principle, one can try to preserve this commutation relation by an appropriate choice of ordering procedure, or by classically vanishing corrections of simplest definition (11). However, now we will show, that anomaly in this

[b] Quasi-classical approximation also gives integer values for T spectrum (T is action-type variable, generating 2π-periodical evolution).

[c] Matrix elements of this operator between states with finite number of excited modes are finite.

[d] because there are anomalous operators under the square root in T, and because we perform redefinitions $T \to [T]$.

commutator is not crucial for the theory, because in spite of this anomaly, we are able to construct the vertex operators, acting in the physical subspace.

Operators $a^\alpha(\sigma)$ themselves do not act in the physical subspace: $a^\alpha(\sigma)|phys\rangle \not\subset |phys\rangle$, due to $[a^\alpha(\sigma), L_0] \neq 0$. Vertex operators are their special combinations. For example, emission of photons from the charged ends of string is described by operator [8]: $V^\alpha = \int_0^{2\pi} \frac{d\sigma}{2\pi} : a^\alpha(\sigma) e^{ikQ(\sigma)} :$ If initial and final states satisfy the mass shell condition $P^2/2\pi - (P-k)^2/2\pi \in \mathbf{Z}$, this operator acts in the physical subspace. Indeed, acting by this operator on the state:

$$V^\alpha |\tfrac{P^2}{2\pi}, \chi_0 = 0\rangle = \int_0^{2\pi} \tfrac{d\sigma}{2\pi} e^{i\chi_0\sigma} : a^\alpha(0) e^{ikQ(0)} : e^{-i\chi_0\sigma} |\tfrac{P^2}{2\pi}, \chi_0 = 0\rangle$$

(here we explicitly extract σ-dependence by embracing evolution operators), we will have the right evolution operator equal to unity, and $\int_0^{2\pi} \frac{d\sigma}{2\pi} e^{i\chi_0\sigma}$ becomes projector to the space with $\chi_0 = 0$. As a result, we obtain physical state $|\frac{(P-k)^2}{2\pi}, \chi_0 = 0\rangle$ ($e^{ikQ(0)}$ includes e^{ikZ} operator, shifting $P \to P - k$).

In analogous way we can define the vertex operator, commuting with the constraint $\Lambda = S - T$: $\tilde{V}^\alpha = \int_0^{2\pi} \frac{d\tau}{2\pi} V^\alpha(\tau)$, $V^\alpha(\tau) = e^{i\Lambda\tau} V^\alpha e^{-i\Lambda\tau}$. The τ-dependence of $V^\alpha(\tau)$ corresponds to the rotations of gauge axis about spin (transformations, generated by Λ), and we average V^α by these rotations. However, in classical mechanics V^α is parametric invariant (constant on τ), consequently, $V^\alpha(\tau) = V^\alpha + f(\tau)$, where all τ-dependent terms are $f(\tau) = O(\hbar)$. Therefore, the constructed vertex operator $\tilde{V}^\alpha$ classically corresponds to the same variable V^α (changes, performed by insertion of evolution operators and averaging are $O(\hbar)$). Now it acts in the physical space. The proof is analogous: $\tilde{V}^\alpha|\Lambda = 0\rangle = \int_0^{2\pi} \frac{d\tau}{2\pi} e^{i\Lambda\tau} V^\alpha e^{-i\Lambda\tau} |\Lambda = 0\rangle$, in the presence of the physical state $e^{-i\Lambda\tau} = 1$, and $\int_0^{2\pi} \frac{d\tau}{2\pi} e^{i\Lambda\tau}$ becomes a projector to $\Lambda = 0$ space.

Also there is an obvious identity $\langle phys'|\tilde{V}^\alpha|phys\rangle = \langle phys'| : a^\alpha(0) e^{ikQ(0)} : |phys\rangle$. Thus, in practical calculations it's sufficient to find matrix elements of operator $: a^\alpha(0) e^{ikQ(0)} :$ between the physical states. Non-physical states do not appear in these calculations.

Conclusion. We have constructed a quantum theory, which acts in 4-dimensional space-time, does not have anomalies in Lorentz group and algebra of constraints, and in classical limit represents Nambu-Goto string theory. We considered 3 variants of this theory:

1. quantization of axially symmetrical world sheets (fig.4, left);
2. quantization of general world sheets by Gupta-Bleuler's procedure (fig.4, right);
3. quantization of general world sheets by Dirac's procedure (fig.5, right).

In the first two cases the quantum theory has no intrinsic difficulties. In the third case there is anomaly in spectrum of an operator T, entering into one of the constraints. To obtain non-empty theory, we should perform classically vanishing corrections of this operator. These corrections are ambiguous, only one variant was considered. Independently on the definition of T, vertex operators can be constructed, acting in the physical subspace.

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Appendix 1: Symplectic structure of the phase space [1,9].

In modern formulation of Hamiltonian mechanics the phase space is defined as a smooth manifold, endowed by a closed non-degenerate differential 2-form $\Omega = \frac{1}{2}\omega_{ij} dX^i \wedge dX^j$ (in some local coordinates X^i, $i = 1, \ldots, 2n$). Poisson brackets are defined by the form as $\{X^i, X^j\} = \omega^{ij}$, where $\|\omega^{ij}\|$ is inverse to $\|\omega_{ij}\|$: $\omega_{ij}\omega^{jk} = \delta_i^k$.

Let's consider a surface in the phase space, given by the 2nd class constraints: $\chi_\alpha(X) = 0$ ($\alpha = 1, \ldots, r$), $det\|\{\chi_\alpha, \chi_\beta\}\| \neq 0$. Reduction on this surface consists in the substitution of its explicit parametrization $X^i = X^i(u^a)$ ($a = 1, \ldots, 2n - r$) into the form:

$$\Omega = \frac{1}{2}\Omega_{ab} du^a \wedge du^b, \quad \Omega_{ab} = \frac{\partial X^i}{\partial u^a}\omega_{ij}\frac{\partial X^j}{\partial u^b}, \quad det\|\Omega_{ab}\| \neq 0.$$

Matrix $\|\Omega^{ab}\|$, inverse to $\|\Omega_{ab}\|$, defines Poisson brackets on the surface: $\{u^a, u^b\} = \Omega^{ab}$.

This method is equivalent to commonly used Dirac brackets' formalism. Sometimes it is convenient to combine both methods: some of the constraints $\chi_\alpha(X)$ are imposed as above, then Dirac brackets on the remaining constraints $\psi_n(u)$ are calculated by definition:

$$\{u^a, u^b\}^D = \{u^a, u^b\} - \{u^a, \psi_n\}\Pi^{nm}\{\psi_m, u^b\},$$

where $\|\Pi^{nm}\|$ is inverse to $\|\Pi_{nm}\|$: $\Pi_{nm} = \{\psi_n, \psi_m\}$.

In string theory canonical Poisson brackets $\{x_\mu(\sigma), p_\nu(\tilde\sigma)\} = g_{\mu\nu}\delta(\sigma - \tilde\sigma)$ correspond to symplectic form $\Omega = \int_0^\pi d\sigma\ \delta p_\mu(\sigma) \wedge \delta x_\mu(\sigma)$. It can be transformed to the form (4) by a substitution of expressions for $x_\mu(\sigma), p_\mu(\sigma)$ in terms of $Q_\mu(\sigma)$; and then to the form (6) by substitution of light cone parametrization for $Q_\mu(\sigma)$.

Spin part of the form $\Omega_S = \frac{1}{2} d\mathbf{e}_i \wedge d(\mathbf{S} \times \mathbf{e}_i)$ corresponds to Poisson brackets (8). To prove this, at first invert coefficient matrix of the form, ignoring orthonormality constraints $\mathbf{e}_i\mathbf{e}_j = \delta_{ij}$, then calculate Dirac's brackets on the surface of these constraints.

Considering *gauge 1*, we substitute $\mathbf{e}_3 = \mathbf{S}/S$, $\mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_1$ into the form and find $\Omega_S = d\mathbf{e}_1 \wedge d(\mathbf{S} \times \mathbf{e}_1)$. Correspondence of this form to Poisson brackets (9) can be proven analogously.

For *gauge 2* we use the following property of symplectic form.

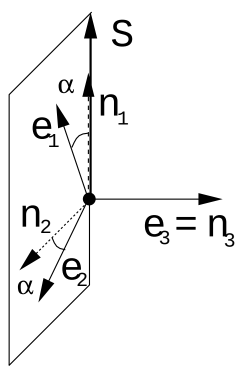

Fig.6. Rotation of the basis for gauge 2.

Lemma. Let $\mathbf{e}_i = R_{ij}^{(k)}\mathbf{n}_j$, where $R_{ij}^{(k)}(\alpha)$ is a matrix of rotation about axis $\mathbf{n}_k$: $R^{(k)} = \exp \alpha \hat{r}^{(k)}$, $\hat{r}_{ij}^{(k)} = \epsilon_{ijk}$. Then $\Omega_S = \frac{1}{2} d\mathbf{n}_i \wedge d(\mathbf{S} \times \mathbf{n}_i) + d\alpha \wedge dS_k$, where $S_k = \mathbf{n}_k\mathbf{S}$ is a projection of spin vector to rotation axis.

Proof. $\Omega_S = \frac{1}{2} d\mathbf{n}_i \wedge d(\mathbf{S} \times \mathbf{n}_i) - \frac{1}{2}(\mathbf{S} \times \mathbf{n}_j, \mathbf{n}_k) dR_{ij}^{(k)} \wedge dR_{ik}^{(k)} + \frac{1}{2}\epsilon_{kjl}\ R_{ik}^{(k)} dR_{ij}^{(k)} \wedge d(\mathbf{n}_l\mathbf{S})$. The second term here is proportional $d\alpha \wedge d\alpha = 0$. Using the identity $(R^{(k)})^T dR^{(k)} = \hat{r}^{(k)} d\alpha$ in the third term, transform it to the form $d\alpha \wedge dS_k$. Lemma is proven.

Let's consider the gauge 2: $\mathbf{e}_3 \perp \mathbf{S}$ (fig.6), and rotate the basis about $\mathbf{e}_3$ to align the first axis along $\mathbf{S}$: $\mathbf{e}_i = R_{ij}^{(3)}(\alpha)\mathbf{n}_j$; $\mathbf{n}_3 = \mathbf{e}_3$, $\mathbf{n}_1 = \mathbf{S}/S$, $\mathbf{n}_2 = \mathbf{n}_3 \times \mathbf{n}_1$; $\cos\alpha = S_1/S$, $\sin\alpha = -S_2/S \Rightarrow \alpha = \arg S^- = \arg A^-$. Because in this rotation $S_3 = 0$, the additional term $d\alpha \wedge dS_3 = 0$. The resulting form can be transformed to $\Omega_S = d\mathbf{n}_3 \wedge d(\mathbf{S} \times \mathbf{n}_3)$, $\mathbf{n}_3 = \mathbf{e}_3$: the top $(\mathbf{e}_i, \mathbf{S})$ is replaced by the rotator $(\mathbf{e}_3, \mathbf{S})$. Replacement $\mathbf{e}_i \to \mathbf{n}_i$ in the expression for configuration (5) transforms it to (10).

Appendix 2. Geometrical reconstruction of the world sheets

Properties 1,2 of supporting curve follow from its definition. Property 3 follows from 4 and 2. Let's prove the property 4:

$$x_\mu(\sigma,\tau) = (Q_\mu(\sigma_1) + Q_\mu(\sigma_2))/2, \quad \sigma_{1,2} = \tau \pm \sigma. \tag{12}$$

This formula was obtained in [4] by direct solution of Hamiltonian equations in Q-representation. Here we will reproduce the proof of this formula in oscillator representation.

Coordinates and momenta of the string are defined by expressions[e]

$$x_\mu(\sigma) = X_\mu + \tfrac{1}{\sqrt{\pi}} \sum_{n\neq 0} \tfrac{a_\mu^n}{in} \cos n\sigma, \quad p_\mu(\sigma) = \tfrac{1}{\sqrt{\pi}} \sum_n a_\mu^n \cos n\sigma,$$

so that formulae (1)-(3) are valid. Poisson brackets for canonical variables:

$$\{a_\mu^n, a_\nu^k\} = in\; g_{\mu\nu}\; \delta^{k,-n}, \quad \{X_\mu, P_\nu\} = g_{\mu\nu}.$$

Hamiltonian of the system is an arbitrary linear combination of Virasoro constraints: $H = \sum c^k L^k$ $(L^k = \sum_n a_\mu^n a_\mu^{k-n},\ c^{k*} = c^{-k})$. Coefficients c^k influence only parametrization of the world sheet. The choice $H = L^0$ corresponds to conformal parametrization (where $(x' \pm \dot{x})^2 = 0$). This Hamiltonian generates phase rotations $a_\mu^n(\tau) = a_\mu^n(0)e^{in\tau}$ and shifts $X_\mu(\tau) = X_\mu(0) + (P_\mu/\pi)\tau$. Using (3), we see that the evolution of function $Q_\mu(\sigma)$ is the shift of its argument: $Q_\mu(\sigma,\tau) = Q_\mu(\tau+\sigma, 0)$. Then, using (2), we have the following evolution for coordinates and momenta:

$$x_\mu(\sigma,\tau) = (Q_\mu(\tau+\sigma,0) + Q_\mu(\tau-\sigma,0))/2, \quad p_\mu(\sigma,\tau) = (Q'_\mu(\tau+\sigma,0) + Q'_\mu(\tau-\sigma,0))/2.$$

Introducing isotropic coordinates $\sigma_{1,2} = \tau \pm \sigma$, obtain formula (12).

Remark: Using Poisson brackets (4), we have $\{Q_\mu(\sigma), Q'^2(\tilde{\sigma})/4\} = \Delta(\sigma - \tilde{\sigma})Q'_\mu(\sigma)$, and for $H = \int d\sigma\ F(\sigma)Q'^2(\sigma)/4$: $\dot{Q}_\mu(\sigma) = \{Q_\mu(\sigma), H\} = F(\sigma)Q'_\mu(\sigma)$, linear combinations of constraints generate shifts of points in tangent direction to the supporting curve, or equivalently – reparametrizations of this curve.

1. Taking $F = 1$, we will obtain the evolution $Q(\sigma) \to Q(\tau+\sigma)$ and formula (12) again.

2. Considering arbitrary F, we will see that the reduced phase space of string (obtained in factorization of the phase space by the action of gauge group) is actually a set of all possible supporting curves, which are considered as geometric images, without respect to their parametrization (two different parametrizations of the curve correspond to the same point of the reduced phase space). All physical observables in string theory are parametric invariants of supporting curve. The world sheet is also reconstructed by the supporting curve in parametrically invariant way, see fig.2.

Appendix 3: Bases of physical subspaces.

Gauge 1, axially symmetrical configurations. Basis of physical subspace is given in Table 1.

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[e] See e.g.[10]. Difference of notations: a_μ^n in our work corresponds to $i\sqrt{n}a_\mu^{n*}$ in [10] $(n > 0)$.

Gauge 1, Gupta-Bleuler's approach. Fig.7 shows the spectrum of operators (L_0, A_3). Due to the constraint $S - A_3 = 0,\ S \geq 0$, we should consider only the upper part of this spectrum: $A_3 \geq 0$.
Operator A^+ acts on the level $L_0 = Const$ and raises A_3 by 1. It annulates all states on leading trajectory $L_0 = A_3$ (among them $|L_0 = 0, A_3 = 0\rangle$ and $|L_0 = 1, A_3 = 1\rangle$). For the states with $0 \leq A_3 < L_0, L_0 \geq 2$ the multiplicity of states on the level $N(L_0, A_3)$ has a property $N(L_0, A_3) \geq N(L_0, A_3 + 1)$. The remaining part of consideration is a proof (done by direct computation), that matrices $A^+_{ij} = \langle L_0, A_3 + 1, i|A^+|L_0, A_3, j\rangle$, representing linear map $A^+ : (L_0, A_3) \to (L_0, A_3 + 1)$, have maximal rank (equal to less dimension $N(L_0, A_3 + 1)$). Therefore, this linear map has a kernel with the dimension $K(L_0, A_3) = N(L_0, A_3) - N(L_0, A_3 + 1)$. (Particularly, the linear map $(3, 1) \to (3, 2)$, marked by arrow on fig.7, has exactly 1-dimensional kernel.) Computing $K(L_0, A_3)$ for the whole spectrum fig.7, we obtain the spectrum fig.4, right. Then we solve the linear equations $A^+_{ij}\Psi_j = 0$, and place the result in Table 2.

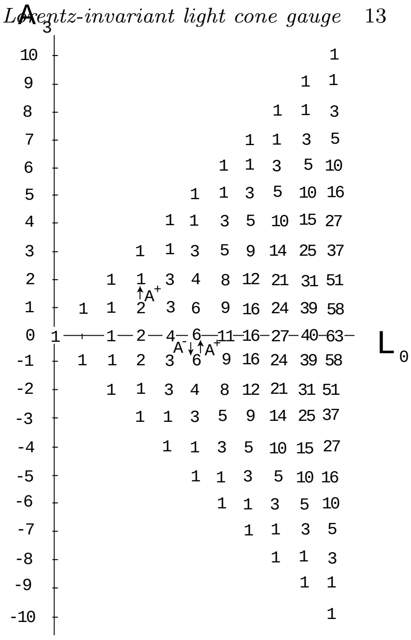

Fig.7. Spectrum (L_0, A_3).

Gauge 2. Operators A^+A^-, A^-A^+ act in subspaces with fixed $(L_0, A_3 = 0)$. Computing matrix elements of these operators, and solving eigenvalue problem for their symmetrical combination $C = (A^+A^- + A^-A^+)/2$, we obtain the spectrum $T_i = \sqrt{C_i}$, shown on fig.5, left. The eigenvectors of C are presented in Table 3.

Π-operation, represented by a replacement $n(k) \to n(-k)$ in state vector, acts in subspaces with fixed $(L_0, A_3 = 0)$ and commutes with C. As a result, all eigenvectors of C have definite Π-parity. Multiplying Π-parity by the factor $(-1)^S,\ S = [T]$, obtain P-parity. Separate spin-mass spectra for the states with definite P-parity are shown on fig.8.

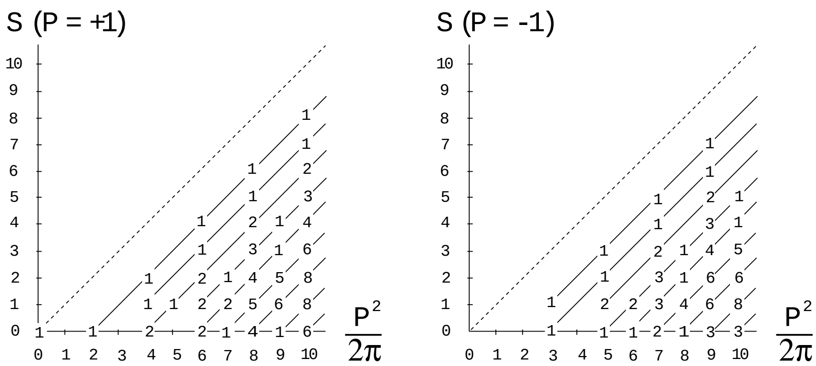

Fig.8. Spin-mass spectra for the states with definite P-parity (gauge 2).

L_0S	$\lvert\{n(k)\}\rangle$	L_0S	$\lvert\{n(k)\}\rangle$	L_0S	$\lvert\{n(k)\}\rangle$
00	$\lvert 0\rangle$	40	$\lvert 2(1)2(-1)\rangle$	51	$\lvert 3(1)2(-1)\rangle$
11	$\lvert 1(1)\rangle$		$\lvert 1(-1)1(3)\rangle$		$\lvert 1(1)1(-1)1(3)\rangle$
20	$\lvert 1(1)1(-1)\rangle$		$\lvert 1(1)1(-3)\rangle$		$\lvert 2(1)1(-3)\rangle$
22	$\lvert 2(1)\rangle$	42	$\lvert 3(1)1(-1)\rangle$		$\lvert 1(5)\rangle$
31	$\lvert 2(1)1(-1)\rangle$		$\lvert 1(1)1(3)\rangle$	53	$\lvert 4(1)1(-1)\rangle$
	$\lvert 1(3)\rangle$	44	$\lvert 4(1)\rangle$		$\lvert 2(1)1(3)\rangle$
33	$\lvert 3(1)\rangle$			55	$\lvert 5(1)\rangle$

Table 1: basis of physical subspace for gauge 1, axially symmetrical configurations.

L_0S	$\|\{n(k)\}\rangle$	L_0S	$\|\{n(k)\}\rangle$
00	$\|0\rangle$	51	$24\|3(1)2(-1)\rangle+9\|1(-1)2(2)\rangle-$
11	$\|1(1)\rangle$		$96\|1(1)1(2)1(-2)\rangle+16\|1(1)1(-1)1(3)\rangle+$
22	$\|2(1)\rangle$		$120\|2(1)1(-3)\rangle;\ 32\|3(1)2(-1)\rangle+7\|1(-1)2(2)\rangle+$
31	$9\|2(1)1(-1)\rangle+2\|1(3)\rangle$		$32\|1(1)1(2)1(-2)\rangle+8\|1(1)1(-1)1(3)\rangle+8\|1(5)\rangle$
33	$\|3(1)\rangle$	52	$-12\|2(1)1(-1)1(2)\rangle+24\|3(1)1(-2)\rangle-$
40	$9\|2(1)2(-1)\rangle-18\|1(2)1(-2)\rangle+$		$4\|1(2)1(3)\rangle+3\|1(1)1(4)\rangle$
	$5\|1(-1)1(3)\rangle+27\|1(1)1(-3)\rangle$	53	$2\|4(1)1(-1)\rangle+\|1(1)2(2)\rangle;$
42	$8\|3(1)1(-1)\rangle+3\|2(2)\rangle;$		$9\|4(1)1(-1)\rangle+4\|2(1)1(3)\rangle$
	$3\|3(1)1(-1)\rangle+\|1(1)1(3)\rangle$	55	$\|5(1)\rangle$
44	$\|4(1)\rangle$		

Table 2: basis of physical subspace for gauge 1, Gupta-Bleuler's approach.

L_0SP	T	$\|\{n(k)\}\rangle$
00+	0	$\|0\rangle$
20+	0.5	$\|1(1)1(-1)\rangle$
30−	0.645	$0.707(\|1(-1)1(2)\rangle-\|1(1)1(-2)\rangle)$
31−	1.48	$0.707(\|1(-1)1(2)\rangle+\|1(1)1(-2)\rangle)$
40+	0.768	$0.630\|2(1)2(-1)\rangle-0.460\|1(2)1(-2)\rangle+$
		$0.442(\|1(-1)1(3)\rangle+\|1(1)1(-3)\rangle)$
40+	0.426	$0.633\|2(1)2(-1)\rangle+0.771\|1(2)1(-2)\rangle-$
		$0.0496(\|1(-1)1(3)\rangle+\|1(1)1(-3)\rangle)$
41+	1.63	$0.707(\|1(-1)1(3)\rangle-\|1(1)1(-3)\rangle)$
42+	2.43	$-0.450\|2(1)2(-1)\rangle+0.440\|1(2)1(-2)\rangle+$
		$0.549(\|1(-1)1(3)\rangle+\|1(1)1(-3)\rangle)$
50−	0.64	$0.424(\|1(1)2(-1)1(2)\rangle-\|2(1)1(-1)1(-2)\rangle)+$
		$0.558(\|1(-2)1(3)\rangle-\|1(2)1(-3)\rangle)+$
		$0.0972(\|1(-1)1(4)\rangle-\|1(1)1(-4)\rangle)$
51+	1.09	$-0.450(\|1(1)2(-1)1(2)\rangle-\|2(1)1(-1)1(-2)\rangle)+$
		$0.406(\|1(-2)1(3)\rangle-\|1(2)1(-3)\rangle)-$
		$0.364(\|1(-1)1(4)\rangle-\|1(1)1(-4)\rangle)$
51−	1.41	$0.210(\|1(1)2(-1)1(2)\rangle+\|2(1)1(-1)1(-2)\rangle)+$
		$0.633(\|1(-2)1(3)\rangle+\|1(2)1(-3)\rangle)-$
		$0.235(\|1(-1)1(4)\rangle+\|1(1)1(-4)\rangle)$
51−	1.61	$0.509(\|1(1)2(-1)1(2)\rangle+\|2(1)1(-1)1(-2)\rangle)+$
		$0.0133(\|1(-2)1(3)\rangle+\|1(2)1(-3)\rangle)+$
		$0.490(\|1(-1)1(4)\rangle+\|1(1)1(-4)\rangle)$
52−	2.62	$0.343(\|1(1)2(-1)1(2)\rangle-\|2(1)1(-1)1(-2)\rangle)-$
		$0.156(\|1(-2)1(3)\rangle-\|1(2)1(-3)\rangle)-$
		$0.598(\|1(-1)1(4)\rangle-\|1(1)1(-4)\rangle)$
53−	3.37	$0.443(\|1(1)2(-1)1(2)\rangle+\|2(1)1(-1)1(-2)\rangle)-$
		$0.315(\|1(-2)1(3)\rangle+\|1(2)1(-3)\rangle)-$
		$0.452(\|1(-1)1(4)\rangle+\|1(1)1(-4)\rangle)$

Table 3: basis of physical subspace for gauge 2.

<u>*Remarks.*</u> In Tables 1,2 $\big|\{n(k)\}\big\rangle = \prod_{k>0}(a_k^+)^{n(k)}(a_{-k})^{n(-k)}\big|0\big\rangle$. Norms of these states can be calculated by the formula $N = \big\langle\{n(k)\}\big|\{n(k)\}\big\rangle = \prod_{k>0} k^{n(k)}n(k)!\ k^{n(-k)}n(-k)!$ In Table 3 the states $\big|\{n(k)\}\big\rangle$ are *normalized*: $\big|\{n(k)\}\big\rangle = N^{-1/2}\prod_{k>0}(a_k^+)^{n(k)}(a_{-k})^{n(-k)}\big|0\big\rangle$. Longer versions of these tables, continued up to values $L_0 = 10$, can be found in Internet under URL: `http://viswiz.gmd.de/~nikitin/str/lcg.html`

Appendix 4: Singularity on leading Regge trajectory.

Let's consider the projection of supporting curve in CMF and parametrize it by it's length: $\mathbf{Q}(L)$. In this parametrization $dQ_0/dL = |d\mathbf{Q}/dL| = 1$ (consequently, total length of the curve is equal to double mass of the string). Let's consider a point on the supporting curve, in which tangent vector $d\mathbf{Q}/dL$ is directed opposite to the gauge axis $\mathbf{e}_3$, see fig.9. In the vicinity of this point the following expansion is valid: $d\mathbf{Q}/dL = \mathbf{T} + \mathbf{N}(L-L^*) + O((L-L^*)^2)$, where $\mathbf{T} = d\mathbf{Q}/dL = -\mathbf{e}_3$

is unit tangent vector to the curve in the point $\mathbf{Q}(L^*)$, and $\mathbf{N} = d^2\mathbf{Q}/dL^2 \perp \mathbf{e}_3$ is a major normal to the curve in this point. Thus, the expansion of $(dQ_0/dL + dQ_3/dL)$ starts from $(L - L^*)^2$.

On the other hand, in light cone gauge (5) $dQ_0/d\sigma + dQ_3/d\sigma = dL/d\sigma\,(dQ_0/dL + dQ_3/dL) = \sqrt{P^2}/\pi = Const$, therefore $dL/d\sigma \sim (L - L^*)^{-2} \;\Rightarrow\; (L - L^*) \sim (\sigma - \sigma^*)^{1/3}$. Then $dQ_0/d\sigma = dL/d\sigma \sim (\sigma-\sigma^*)^{-2/3}$, $dQ_3/d\sigma = \sqrt{P^2}/\pi - dQ_0/d\sigma \sim (\sigma-\sigma^*)^{-2/3}$, and $(dQ_1/d\sigma)^2 + (dQ_2/d\sigma)^2 = 2\sqrt{P^2}/\pi \cdot dQ_0/d\sigma - P^2/\pi^2 \sim (\sigma - \sigma^*)^{-2/3}$.

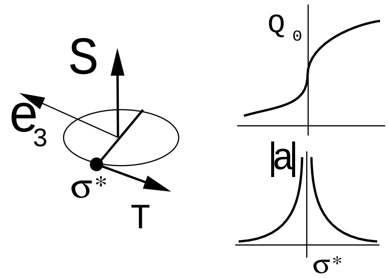

Fig.9. Singularity of light cone gauge.

Thus, the light cone gauge parameterizes the supporting curve irregularly in the vicinity of point $Q_\mu(\sigma^*)$. At $\sigma \to \sigma^*$ the components of tangent vector $Q'_\mu(\sigma)$ tend to infinity: $Q'_{0,3}(\sigma)$ as $(\sigma-\sigma^*)^{-2/3}$, and $Q'_{1,2}(\sigma)$ as $(\sigma - \sigma^*)^{-1/3}$. This type of singularity is integrable, it does not create any problems for classical theory. Particularly, masses of such configurations are finite. However, such configurations require an infinite number of excited modes in Fourier expansion of $a(\sigma) = Q'_1(\sigma) - iQ'_2(\sigma)$. This makes problematic a consideration of such configurations in quantum theory, where the states with finite mass necessarily have finite number of filled modes.

For string theory in $d = 4$ the described singularity is not crucial, because the alignment of $\mathbf{Q}'(\sigma^*) \uparrow\downarrow \mathbf{e}_3$ can be removed by a small deformation of the curve, so the configurations with such singularity are rare in the whole phase space. (This singularity is a real obstacle for $d = 3$. In 2-dimensional CMF such alignment cannot be removed by a small deformation, and most of string configurations become infinite-modal in light cone gauge.)

In the case, if the gauge axis is selected perpendicular to the spin (gauge 2), the described singularity appears on the leading Regge trajectory. Leading trajectory corresponds to circular supporting curves (straight strings [1]), and for any direction of the gauge axis, perpendicular to the spin, a point on the circle exists with tangent opposite to the gauge axis. This singularity is a possible reason for disappearance of the leading trajectory in quantization.

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